PROGRESS IN LEARNING ALGEBRA: TEMPORARY AND PERSISTENT DIFFICULTIES

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The research reported in this paper is part of a large study[1] *investigating the cognitive and linguistic aspects of learning algebra in secondary school. Five of the 22 schools participating in the study tested the same cohort of students on two or three occasions. Analysis of the data from these five schools has shown that certain initial difficulties are easily corrected for the majority of students, whereas others are more persistent. Most students by Years 9 and 10 had reached an understanding of algebraic letters as unknown numbers, but their understanding of equations was less secure. Major persistent difficulties were the significance of brackets and the notation for products and powers. A small proportion of students made very little or no progress.*

It is widely acknowledged, both locally and internationally, that the majority of students learn algebra with great difficulty. Large-scale studies such as the Assessment of Performance Unit [APU] (1985), the Concepts in Secondary Mathematics and Science [CSMS] programme (see Kuchemann, 1981), the International Educational Achievement [IEA] study (see Robitaille & Garden, 1989) and the National Assessment of Educational Progress [NAEP] reports (see Herscovics, 1989) have all shown that students' achievement is low on elementary algebra items. Possible causes of misunderstanding and failure that have been suggested include students' level of cognitive development, inadequate understanding of number properties and the operations of arithmetic, and the intrinsic difficulty of learning a formal notation system. Kuchemann (1981), for example, interpreted his research results within a framework of cognitive development. He concluded that although students made steady progress, at age 15 no more than 40% had reached a cognitive level where they could deal with algebraic letters as unknown numbers or variables. Also within a developmental framework, a longitudinal study by Booth (1984) found evidence for a maturation-linked factor in students' abilities to (a) understand the meanings of algebraic letters, and (b) recognise and formalise mathematical methods. However, as Booth pointed out, cognitive development in itself does not ensure the growth of these abilities; appropriate learning experiences are required.

The focus of this paper is students' growth in understanding of some fundamental ideas of algebra, as shown by their performance in a series of tests.

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STUDENTS' INTERPRETATION OF TEST ITEMS

Wrong responses to test items have a wide variety of causes. Both Booth (1984) and Chalouh and Herscovics (1988) refer to the influence of the student's frame of reference or the context of a problem on the "answer" that the student chooses. Another factor in certain items is the student's uncertainty about what is required. For example, students may often hear teachers say that "you can't add unlike terms" but then in a test are instructed: "Add 4 onto $3n$" (Kuchemann, 1981, p. 108). Since these are unlike terms, how can they be added? Similarly, when told "x is any number; write the number which is 3 more than x" (Bell, Costello & Kuchemann, 1983, p. 138), the large number of students who gave x an arbitrary value and added 3 to it may have thought that since x is "any number" that is what they were supposed to do. Other wrong responses reflect students' assumptions, developed from arithmetic, about what an acceptable "answer" should look like (e.g., that it should not contain an operation sign). In the present study all these factors were taken into account as far as possible both in the design of items and the classification of responses.

SELECTION OF SAMPLE AND TESTING PROCEDURE

Algebra items were prepared by the researchers and included in tests used in 22 Victorian secondary schools in 1991-93. In another school, individual students were interviewed. After the first test was used in 1991, several of the teachers wanted to carry out further testing in 1992 and 1993. The results of the repeated testing form the data for this paper, as shown in Figure 1. The versions of the tests were not identical. New items were inserted in Tests 2 or 3 to probe more deeply certain errors that had been widespread in the first test, and some original items were re-worded to avoid misinterpretation or were extended to explore ways of prompting correct responses. School A and the school where students were interviewed are state schools in low socio-economic areas of Melbourne. The other four schools, two Catholic and two non-Catholic, are in middle-class suburbs.

School	Year	n	Test 1	Test 2	Test 3
A (Co-ed)	7	65	Mar 92	June 92	-
B (Girls)	8	52	-	July, Sep 92	-
C (Boys)	9	54	Mar 92	Oct 92	-
D (Girls)	8-9	87	May 91	Nov 91	June 92
E (Boys)	8-9	81	May 91	Nov 91	June 92
C (Boys)	9-10	27	Mar 92	Oct 92	Apr 93

Figure 1. Year levels, tests and dates when tested

DISCUSSION OF RESULTS

Details of all test items and students' performance are too extensive to summarise in this paper, and may be obtained from the authors. We present here a brief overview of students' developing competence in four essential basic algebraic skills:

- recognising what operation relates two quantities;
- using algebra notation to write an expression;
- interpreting an equation;
- writing an equation.

1. Recognising an operation

Except for early in Year 7, most students were able to recognise the operation relating two quantities in familiar contexts such as combining sections of a journey (addition) or sharing money among a number of people (division). At all levels a few students were not sure whether to choose multiplication or division in less familiar situations (e.g., how to work out the number of rows of seats in a theatre, given 500 seats and 25 per row). Many students are uncertain about the correct order of terms in division (e.g., 500÷25 or 25÷500) or they think it is not important. This uncertainty or lack of concern for order of terms affected many students' success in several of the algebraic items when they were first tested, but improved over time.

Factors that affected students' choices of operation included:

- salient and misleading aspects of a problem (e.g., the opposite directions on a signpost, implying negative and positive values, when distances have to added);
- large numbers, preventing a mental image of the situation and judgment of an answer as reasonable;
- students' tendency to work out subtraction problems by "adding on", and division problems from known products.

All these factors should be taken into account when teachers are dealing with non-algebraic sections of the curriculum. Algebra learning needs a firm foundation of number knowledge and a confident familiarity with the four operations.

2. Using algebraic notation to write an expression

Three well-documented obstacles to the use of algebraic notation are:

- the meanings of letters;
- the belief that an expression containing an operation sign should be simplified to a single "answer" without an operation sign;
- lack of awareness of the need for brackets.

Several items in the tests were used to assess progress in overcoming these obstacles.

Students begin learning algebra with considerable experience of letters as abbreviated words (e.g., *h* means *height*) and as coded numbers (A stands for 1, B stands for 2, etc.) and as they start algebra they use letters to stand for unknown numbers. In the first test, the majority of Year 7 responses to simple algebra items were arbitrary numbers. For example, popular answers for DAVID'S HEIGHT (see below) were 20, 90 and 110. In later tests, and in all other year levels, very few students did this. Some students in Year 8 and 9 classes assumed that algebraic letters are coded numbers, having values according to their positions in the alphabet (e.g., $h = 8$, therefore $h+10 = 18 = r$). This misunderstanding was easily corrected by teachers. Only one Year 10 student continued to make the error.

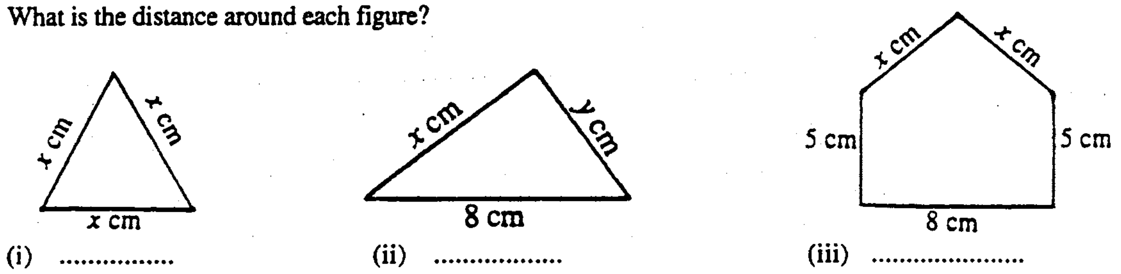
David is 10 cm taller than Con. Con is *h* cm tall.
Use algebra to write David's height. ....

Many students at all levels who initially had omitted algebraic items, given arbitrary numerical answers, assumed alphabetical coding, or used letters to stand for words, progressed to writing algebraic expressions that, although not always correct, used letters to signify unknown numbers. The Year 7 group, for example, made good progress in learning how algebraic letters might be used. In Test 1, most of them had chosen a value for a letter. This behaviour is in accord with the view of Kuchemann and others that students who have not reached a certain cognitive level are unable to use letters without evaluating them first. However after three months only about 15% of the group chose numerical values for letters, and very few used letters to stand for words. Most students made reasonable attempts at using algebraic notation (e.g., h -10 or $10h$ for DAVID'S HEIGHT), and one-third were correct on most algebra items. Their performance was impressive, and indicates the effectiveness of the teaching program that had been used.

A new misinterpretation of algebraic letters appeared in older classes. This was the assumption that the value of any variable is 1. The fact that this error was not widespread and occurred in only two of the schools suggests that it may be caused by students misinterpreting teachers' explanations. As we have mentioned earlier, students sometimes misinterpret "x is any number". It appears that when teachers stress that "x without a coefficient is $1x$ " they may also be misunderstood, since some students interpret this to mean "x by itself is one". Another possible source of confusion is the fact that in the context of indices the power of x is 1 if no index is written (i.e., $x = x^1$). When interviewed, several Year 10 students explained, "By itself x is 1"; "It's 1 because it hasn't got a number"; "x is just one single thing, so x times x is just like 1 times 1"; and "$8+x+y$ equals 10 because x and y are equal to 1".

Writing algebraic expressions was difficult in a geometrical context, as in the item below (adapted from Kuchemann, 1981). Forms of this item have been extensively used in studies of algebraic thinking. The Kuchemann version of (i),with each side labelled "e" where we have used "x cm", was answered correctly by 94% of 14-year-olds in the CSMS study (Kuchemann, 1981, p. 102). Our version of (i) was considerably harder. This may be because the letter x is not perceived as standing for anything concrete, whereas students could interpret "e" as "edge" and write the correct answer 3e which for them meant "three edges".

What is the distance around each figure?



Low success rates (not more than 30% correct) in Year 7 and early Year 8 were mainly due to the large proportion of numerical substitutions and omissions. Some students thought they were supposed to measure the lines. At higher year levels, success rates improved on repeated testing reaching about 60% in two schools (similar to the Kuchemann result for the same age group on part (iii)) and 80% in one school. The cause of many errors, in all three parts of the item, was the incorrect use of exponential notation (e.g., x^3 instead of $3x$), which is not referred to in Kuchemann's report. There appear to be many factors contributing to the difficulty of the item which are not specifically associated with algebraic thinking or the meanings of letters. When students were interviewed, they made comments such as "Do I have to figure out the numbers?", "That's the hypotenuse", or "If it's 8 across that way, then rule off the line and cut straight up", indicating that they were searching for remembered schemas or learned procedures from geometry. These and other comments suggest that students did not have a clear conceptual model of the task as requiring only a sum of three measures. It was interesting to see that for part (iii) several Year 10 students wrote $x^2 + 5^2 + 8$. Their uncertainty about how to write "twice x" or "x plus x" (discussed below) has contaminated their knowledge of how to write "twice five".

The high incidence of answers such as x^3 and $8xy$ for the first parts of the item suggests that students may have been thinking of area and trying to represent a multiplication. To find out whether this was so or whether they think x^3 means "$x + x + x$" we used the item below for Years 8 to 10.

Which of the following expressions can be written as $x + x + x + x$? (Circle one or more of the answers below)

$x + 4$ $\quad$ $x \times 4$ $\quad$ $4x$ $\quad$ x^4 $\quad$ 4^x

Although "collecting terms" is one of the first procedures students learn in beginning algebra, approximately one-third of students were not correct. The most popular wrong choice was x^4. Many students who circled x^4 also circled $x \times 4$. Some circled both x^4 and 4^x. The responses indicate confusion about the concepts of adding, multiplying and exponentiation, as well as about the notation for these procedures, which did not improve over the time of the testing in the three schools involved (remaining at around 60% correct in years 8, 9 and 10).

The following item required students to coordinate two operations, and very few students at any level were correct.

> Write the following in mathematical symbols.
> "Add 5 to an unknown number x, then multiply the result by 3". ....

After seeing the very poor results from the first test, which ranged from 8% at Year 7 up to 35% at Year 9, we prepared a lesson module on the use of brackets for ordering operations, to be used by Year-9 teachers in one school. Their efforts to teach the use of brackets were clearly very effective. In the first test only 25% had been correct. In the second test, given six months later, more than 70% were correct. We conclude that many of the students had learned how to express a sequence of operations and the purpose of brackets, either from the lesson module or some other source, since the previous test. However in a similar item given the following year to some of the same students the success rate dropped to 60%, suggesting that for some students their knowledge of the purpose of brackets had not been practised or put to use and was consequently forgotten.

3. Interpreting an equation

The following item was used with Years 8, 9 and 10.

> a and b stand for numbers (but not zero). We know that $a = 37 + b$.
> Which of the following must be true? (Tick one)
> (i) a is greater than b (iii) $a = 37$
> (ii) b is greater than a (iv) you can't tell which number is greater.

Approximately one-quarter of students at all levels made incorrect choices. There was no evidence of improved understanding from one test to the next. In all groups by far the most common wrong choice was "You can't tell". In some classes, the proportion of "You can't tell" responses increased on repeated testing. The results suggest that about 25% students, even at Year 10 level, have an unstable concept of what an equation is in different contexts. Further evidence of this instability is found from the following item, presented in two versions:

Early version

> I have x dollars and you have y dollars. I have $6 more than you.
> Which of the following equations must be true? (Underline one of the answers below)
> $6y = x$ $6x = y$ $y + 6 = x$ $x + 6 = y$ $x - 6 = y$

Later version

> In a class there are six more boys than girls.
> To find the number of girls, would you
> (i) Add 6 to the number of boys? (ii) Subtract 6 from the number of boys? (Tick one)
> If we write p for the number of girls and s for the number of boys, which of the following are correct? (Underline any that are correct)
> $s + 6 = p$ $p + 6 = s$ $s > p + 6$ $s - 6 = p$ $6p = s$ $6s = p$

On the early version, success rates had ranged from 30% at Year 7 to a maximum of about 60% at Year 9. Only one of the six groups in Figure 1 improved on this item by more than 4% from Test 1 to Test 2. Schools D and E were then given the later version in Test 3. Their success rates from Test 2 to Test 3 increased from 43% to 86% for one group, and from 62% to 83% for the other, suggesting that the hint to look at the equation in a procedural way was very effective. However there was no automatic transfer of their understanding of the "boys and girls" equation to the "a=37+b" item (see above) which appeared later in the same test. The performance of both groups got worse for this item on repeated testing, with increasing numbers of students choosing "You can't tell". We have no explanation as yet for this trend.

4. Writing an equation

Students were required to generate a formula relating x and y from the following table. The questions asking them to read off or calculate values, and to explain their rule in English, were intended to help them think about the relationship and hence construct the correct formula. Although most of them could use the information given in the table for calculating further values, many could not write an equation.

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	11
8	..
..	..

(i) When x is 2, what is y? ....

(ii) When x is 8, what is y? ....

(iii) When x is 800, what is y? ....

(iv) Explain in words how to work out y if you are told what x is ....
....

(v) Use algebra symbols to write a rule connecting x and y. ....

In many schools at present, writing equations from patterns and tables is one of the first algebraic tasks students are expected to master in Years 7 or 8. These learning experiences have clearly not been effective for many students in the sample. Many students did not know what an equation should look like. For example, responses from the Year 10 group included $x+4y$; x to y; 1,5; $x+y$; xy and $x=0,y=4$. There was no improvement in the Year 8-9 and 9-10 groups, where success rates remained steady or declined.

CONCLUSIONS

Our findings indicate that, when only one operation is involved, some students in Year 7 and more than half the students in Years 8 and above can read and understand a simple problem, select the operation, and use an algebraic letter as "unknown number" to write an expression. On almost all the test items there was steady improvement, with some rapid gains in year 7 on the recognition of operations and the use of algebraic notation. Four persistent difficulties over Years 7 to 10 were:

- use of brackets;
- formulating an equation from a table of values;
- using algebraic notation to write repeated addition.

Our findings confirmed that the majority of students at all levels do not remember to use brackets or do not know how they affect the meaning of an expression. A teaching intervention was highly successful in the short term, with some long-term retention. Students' persistent misuse of exponents suggests an insecure foundation of the concepts of multiplication, repeated addition and repeated multiplication. It is shocking that only 60% of the year 9 and 10 students knew that $x + x + x + x$ can be written as $4x$, considering that they have been "doing algebra" for two or three years. Students' interpretation of an equation was greatly assisted when they were asked to make a judgment between two procedural meanings. However when asked to recognise a procedure and write about it ("Explain in words how to work out y"), they had difficulty in formulating the corresponding equation. Far too many students in Years 9 and 10 perceived no relationship between two variables in an equation. Certain incorrect use of algebraic notation appears to be caused by students misunderstanding teachers' explanations; there was no opportunity in the present study to follow the progress of the students concerned. However the belief that the value of a letter depends on its position in the alphabet, particularly common in one school, was easily remedied (as shown by subsequent testing) when brought to the notice of teachers.

A few students appeared to make little or no progress and continued to make the same fundamental errors in Years 9 and 10 that are made by beginners. For the majority, over the period of the testing there was general improvement on most items, particularly in Years 7 and 8. However the prevalence of intuitive responses, guesses and invented notation indicates that students do not have sufficient opportunity to practise using algebra as a precise and meaningful language.

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